Some theorems and remarks on interpolation.

By P. ERDŐS in Syracuse, N. Y.

Throughout this paper $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ will denote the roots of the n-th Chebyshev polynomial $T_n(x)$ $[T_n(\cos\vartheta) = \cos n\vartheta]$. $f(x)$ will denote a function continuous in $[-1, +1]$ and $L_n(f(x))$ will denote the Lagrange interpolation polynomial of $f(x)$ taken at the points $x_i^{(n)}$ $(i = 1, 2, \ldots, n)$; in other words, $L_n(f(x))$ is a polynomial of degree not greater than $(n-1)$ for which[1])

$$L_n(f(x_i^{(n)})) = f(x_i^{(n)}) \qquad (i = 1, 2, \ldots, n).$$

It is a well known result[2]) that for every x_0 there exists a continuous[3]) $f(x)$ so that $L_n(f(x_0))$ does not converge to $f(x_0)$. In fact I proved[4]) that in marked contrast to a well known theorem of FEJÉR on Fourier series, if $x_0 = \cos\frac{p}{q}\pi$, $p \equiv q \equiv 1 \pmod 2$, there exists a continuous $f(x)$ so that $\lim_n |L_n(f(x_0))| = \infty$, or there does not exist a sequence n_i so that $L_{n_i}(f(x_0)) \to f(x_0)$. Moreover[4]) for any $-\infty \leqq c \leqq \infty$ and any such value of x_0 there exists a continuous $f(x)$ so that $L_n(f(x_0)) \to c$, i. e. $L_n(f(x_0))$ can converge to any prescribed value. TURÁN and I[5]) showed that if $x_0 \neq \cos\frac{p}{q}\pi$, $p \equiv q \equiv 1 \pmod 2$, then for every continuous $f(x)$ there exists a sequence n_i so that $L_{n_i}(f(x_0)) \to f(x_0)$. Previously MARCINKIEWICZ[6]) has showed that, if the $x_i^{(n)}$ are the roots of $U_n(x)$,

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[1]) There will be no misunderstanding writing $L_n(f(x_i^{(n)}))$ instead of $L_n(f(x))_{x=x_i^{(n)}}$ throughout this paper.

[2]) For $x_0 = 0$ see L. FEJÉR, Über Interpolation, *Göttinger Nachrichten*, 1916, pp. 1–16. For every x_0 this has been remarked apparently first by S. BERNSTEIN: see his paper "Sur la limitation des valeurs etc.", *Bulletin Acad. Sci. URSS*, 1931, pp. 1025–1050.

[3]) "Continuous" means throughout this paper "continuous in $[-1, +1]$".

[4]) P. ERDŐS, On divergence properties of the Lagrange interpolation parabolas, *Annals of Math.*, **42** (1941), pp. 309–315; P. ERDŐS, Corrections to two of my papers, *ibidem*, **44** (1943), pp. 647–651.

[5]) P. ERDŐS and P. TURÁN, On interpolation. I., *Annals of Math.*, **38** (1937), pp. 142–155.

[6]) J. MARCINKIEWICZ, Sur l'interpolation, *Studia Math.*, **6** (1936), pp. 1–17.

then to a given continuous $f(x)$ and $-1 \leqq x_0' \leqq 1$ there always exists a subsequence n_i, with $L_{n_i}(f(x_0')) \to f(x_0')$. It follows from results of TURÁN and myself[5]) that for a general class of point groups [which include the roots of both $T_n(x)$ and $U_n(x)$], to every continuous $f(x)$ there exists a sequence n_i so that $L_{n_i}(f(x))$ converges to $f(x)$ almost everywhere.

GRÜNWALD and MARCINKIEWICZ[7]) showed that there exists a continuous $f(x)$ so that $L_n(f(x))$ diverges for every x, in fact $\limsup L_n(f(x)) = \infty$ for every x. The analogous question for Fourier series is as well known still unsolved and seems very difficult. MARCINKIEWICZ[6]) and TURÁN[4]) showed that for every x_0 there exists a continuous $f(x)$ so that

$$\limsup \frac{1}{n} \sum_{k=1}^{n} L_k(f(x_0)) = \infty.$$

In other words the arithmetic means of the Lagrange interpolation polynomials of a continuous function can diverge at a given point, again in marked contrast to the celebrated theorem of FEJÉR for Fourier series. MARCINKIEWICZ[6]) further showed that there exists a continuous $f(x)$ so that $\frac{1}{n} \sum_{k=1}^{n} L_k(f(x_0))$ diverges in an arbitrarily given countable set, and he raised the problem whether there exists a continuous $f(x)$ so that $\frac{1}{n} \sum_{k=1}^{n} L_k(f(x_0))$ should diverge almost everywhere. In a paper written 12 years ago G. GRÜNWALD and I[8]) proved that there exists a continuous $f(x)$ so that $\frac{1}{n} \sum_{k=1}^{n} L_k(f(x))$ diverges for every x. While writing this paper I made the unfortunate discovery that our proof is erroneous. All that we prove is that there exists a continuous $f(x)$ so that for every x

$$\limsup \frac{1}{n} \sum_{k=1}^{n} |L_k(f(x))| = \infty. \tag{1}$$

In a view of the strong convergence of the arithmetic means of the Fourier series (1) seems not uninteresting, but of course it would be of interest to know whether (1) holds without the absolute value. I just succeeded to show that with a slight modification of our construction one can prove the existence of a continuous $f(x)$ so that for almost all x

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7) G. GRÜNWALD, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen, *Annals of Math.*, **37** (1936), pp. 908–918; J. MARCINKIEWICZ, Sur la divergence des polynomes d'interpolation, *these Acta*, **8** (1937), pp. 131—135.

8) P. ERDŐS and G. GRÜNWALD, Über die arithmetischen Mittelwerte der Lagrangeschen Interpolationspolynome, *Studia Math.*, **7** (1938), pp. 82—95. The error occurs in the last formula of p. 92.

$$\limsup \frac{1}{n}\left(\sum_{k=1}^{n} L_k(f(x))\right)=\infty. \tag{2}$$

The proof of (2) will be given at another occasion. At present I can not decide whether (2) can diverge for *every* x.

It is easy to prove that if for every x, $|f(x+h)-f(x)|=o\left(\left(\log\frac{1}{|h|}\right)^{-1}\right)$. $L_n(f(x))$ converges to $f(x)$. MARCINKIEWICZ[6]) proved that there exists an $f(x)$ satisfying for every x the inequality $|f(x+h)-f(x)|=O\left(\left(\log\frac{1}{|h|}\right)^{-1}\right)$ such that $L_n(f(x))$ diverges almost everywhere. This result is interesting since it is easy to see that $|L_n f(x)|$ is uniformly bounded in this case. By using the method of GRÜNWALD[7]) it is easy to construct an $f(x)$ satisfying uniformly the "logarithmic" Lipschitz-condition $|f(x+h)-f(x)|=O\left(\left(\log\frac{1}{|h|}\right)^{-1}\right)$ and $L_n(f(x))$ diverges for *every* x. The proof follows the GRÜNWALD—MARCINKIEWICZ ideas closely, thus we do not give it.

In the present paper we prove the following

Theorem 1. *For almost all x*

$$\frac{1}{n}\sum_{k=1}^{n}|L_k(f(x))|=o(\log\log n),$$

if only $f(x)$ is continuous in $[-1,+1]$.

As an easy consequence of Theorem 1 we deduce

Theorem 2. *Let $|f(x+h)-f(x)|=o\left(\left(\log\log\frac{1}{|h|}\right)^{-1}\right)$ uniformly in $[-1,+1]$. Then for almost all x*

$$\frac{1}{n}\sum_{k=1}^{n}(f(x)-L_k(f(x)))^2\to 0.$$

The interest of Theorem 1 and 2 is that they show that taking arithmetical means improves to some extent the convergence-properties of the Lagrange-interpolatorical polynomials.

It can be shown that Theorems 1 and 2 are best possible in the following sense: Let $g(n)\to\infty$ arbitrarily slowly. Then there exists a continuous $f(x)$ so that for almost all x there exists a sequence $n_i=n_i(x)$ so that

$$\frac{1}{n_i}\sum_{\nu=1}^{n_i}|L_\nu(f(x))|>\frac{\log\log n_i}{g(n_i)}. \tag{3}$$

The same holds if we consider arithmetic means of higher order. (3) probably remains true without the absolute value, but this I can not prove.

Also there exists an $f(x)$ satisfying in $-1\leqq x\leqq+1$ the condition

$|f(x+h)-f(x)|=O\left(\left(\log\log\frac{1}{|h|}\right)^{-1}\right)$ and so that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(f(x)\,L_k(f-(x)))^2=0 \tag{4}$$

holds only in a set of measure 0. This is not without interest since in (4) the lim sup is finite for almost x. [This can be shown by the same method as Theorem 2.] The proof of (3) and (4) is fairly complicated but is similar to the [correct part] of the argument of GRÜNWALD and myself.

MARCINKIEWICZ[6]) proved that for every $g(n)\to\infty$ there exists a continuous $f(x)$ so that for almost all x there exists a sequence $n_i=n_i(x)$, for which

$$L_{n_i}(f(x))>\frac{\log n_i}{g(n_i)}.$$

By using the method of GRÜNWALD[7]) it is easy to replace "almost all" by "all" in the result of MARCINKIEWICZ. Further for every x_0 there exists a continuous $f(x)$ so that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}L_k(f(x_0))>\frac{\log n_i}{g(n_i)}.$$

We do not give the proofs of these two theorems since they do not contain any new idea.

Proof of Theorem 1. It will be sufficient to prove that if $f(x)$ is bounded and e. g. $|f(x)|\leqq\frac{1}{2}$ uniformly, then exists an absolute constant c so that for almost all x and $n>n_0=n_0(x)$

$$\frac{1}{n}\sum_{k=1}^{n}|L_k(f(x))|<c\log\log n. \tag{5}$$

Suppose (5) is proved. According to the theorem of WEIERSTRASS we find a polynomial of degree l, $\varphi_l(x)$ so that $|f(x)-\varphi_l(x)|<\varepsilon$. By applying (5) to $(f(x)-\varphi_l(x))$ and remarking that for $k>l$ $L_k(\varphi_l(x))=\varphi_l(x)$ we obtain

$$\frac{1}{n}\sum_{k=1}^{n}|L_k(f(x))|\leqq\frac{1}{n}\sum_{k=1}^{n}|L_k(\varphi_l(x))|+\frac{1}{n}\sum_{k=1}^{n}|L_k(f(x)-\varphi_l(x))|<$$
$$<c\varepsilon\log\log n+O(1)=o(\log\log n),$$

which proves Theorem 1. Thus we only have prove (5).

In the proof of (5) our principal tool will be the following result of MARCINKIEWICZ and ZYGMUND[9]): Let $|f(x)|\leqq 1$, then there exists an absolute constant λ so that for every $a<\lambda$

$$\int_{-1}^{+1}\exp|aL_k(f(x))|\,dx<A=A(\lambda), \tag{6}$$

[9]) J. MARCINKIEWICZ and A. ZYGMUND, Mean values of trigonometrical polynomials, *Fundamenta Math.*, **28** (1937), pp. 131—166, see theorem 4 on p. 133.

where A is independent of k. From (6) we obtain that there exists a constant c_1 so that

(7) $$M[x; |L_k(f(x))| > c_1 \log\log k] < \frac{1}{(\log k)^{10}}$$

($M[x; \ldots]$ denotes the measure of a set in x satisfying a certain condition). Further it is well known that

(8) $$|L_k(f(x))| < c_2 \log k.$$

Now we prove the following

Lemma. *Let $g_k(x) \geqq 0$ be defined in $[-1, +1]$. Assume that*

(9) $$M[x; g_k(x) > c_1 \log\log k] < \frac{1}{(\log k)^{10}}$$

and

(10 $$g_k(x) < c_2 \log k \qquad (k = 2, 3, \ldots, n).$$

Then if c_3 is sufficiently large, we have for almost all x and sufficiently large $n > n_0(x)$

$$G_n(x) = \frac{1}{n} \sum_{k=1}^{n} g_k(x) < c_3 \log\log n.$$

The sequence $|L_k(f(x))|$ satisfies (9) and (10) [since it satisfies (7) and (8)]. Thus to prove Theorem 1 it will suffice to prove our lemma.

Proof of the Lemma. Define $S_k(x)$ as the set in x for which $g_k(x) > c_1 \log\log k$.

Let $\varphi_k(x)$ be 1 in $S_k(x)$ and 0 elsewhere. It follows from (10) that if $G_n(x) > c_1 \log\log n$ then x must be in $S_k(x)$ for at least $c_4 n/\log n$ values of k $(1 \leqq k \leqq n)$. Thus by (9)

(11) $$M[x; G_n(x) > c_1 \log\log n] \leqq M\left[x; \sum_{k=1}^{n} \varphi_k(x) > \frac{c_4 n}{\log n}\right] < \\ < \frac{\log n}{c_4 n} \int_0^1 \left(\sum_{k=1}^{n} \varphi_k(x)\right) dx < \frac{\log n}{c_4 n} \sum_{k=1}^{n} \frac{1}{(\log k)^{10}} < \frac{1}{(\log n)^5}.$$

Put $m_r = [e^{\sqrt{r}}]$. We obtain from (11) that

$$\sum_{r=1}^{\infty} M[x; G_{m_r}(x) > c_1 \log\log m_r] < \sum_{r=1}^{\infty} \frac{1}{r^{5/2}} < \infty.$$

Hence by a simple argument we obtain that for almost all x and large enough $r > r_0(x)$

(12) $$G_{m_r}(x) < c_1 \log\log m_r.$$

If $m_r < n < m_{r+1}$ we have by (10)

$$m_r |G_n(x)| < n |G_n(x)| < m_r |G_{m_r}(x)| + c_2(m_{r+1} - m_r) \log n < m_r |G_{m_r}(x)| + \frac{c_5 m_r}{\sqrt{r}} \log n$$

i. e.

(13) $$|G_n(x)| < |G_{m_r}(x)| + O(1).$$

The Lemma now follows from (12) and (13). Thus Theorem 1 is proved.

Sketch of the proof of Theorem 2. It is well known that there exists a polynomial $\varphi_l(x)$ of degree $< \sqrt{n}$ so that

(14) $$|f(x) - \varphi_l(x)| = o\left(\frac{1}{\log\log n}\right).$$

We have

$$\frac{1}{n}\left[\sum_{k=1}^{n}(f(x)-L_k(f(x)))^2\right] = \frac{1}{n}\left[\sum_{k=1}^{n}(f(x)-L_k(f(x)-\varphi_l(x))-L_k(\varphi_l(x)))^2\right] \leqq$$

(15) $$\leqq \frac{1}{n}\sum_{k=1}^{n}(f(x)-L_k(\varphi_l(x)))^2 + \frac{2}{n}\left[\sum_{k=1}^{n}(f(x)-L_k(\varphi_l(x)))\cdot(L_k(f(x)-\varphi_l(x)))\right] +$$

$$+\frac{1}{n}\sum_{k=1}^{n}L_k(f(x)-\varphi_l(x))^2 = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

From (14) and (8) we obtain $\Sigma_1 = o(1)$ since $L_k(\varphi_l(x)) = \varphi_l(x)$ for $k > \sqrt{n}$. From (14) and (8) we have

$$\Sigma_2 < \frac{\varepsilon}{n\log\log n}\sum_{k=1}^{n}|L_k(f(x)-\varphi_l(x))| + o(1) <$$

$$< \frac{\varepsilon}{n\log\log n}\left(\sum_{k=1}^{n}|L_k(f(x))| + \sum_{k=1}^{n}|L_k(\varphi_l(x))|\right) + o(1) <$$

$$< \frac{2\varepsilon}{n\log\log n}\sum_{k=1}^{n}|L_k(f(x))| + o(1) = o(1)$$

for almost all x by Theorem 1. Now we investigate Σ_3. Put $s_r = e^{r^{1/4}}$. It suffices to show that for almost all x

(16) $$\Sigma_3' = \frac{1}{s_r}\sum_{1\leqq k\leqq s_r}L_k(f(x)-\varphi_l(x))^2 = o(1).$$

For if (16) is proved, put $s_r < n < s_{r+1}$. Clearly

$$\Sigma_3 \leqq \Sigma_3' + \frac{2}{n}\left[\sum_{k=s_r}^{s_{r+1}}L_k(f(x)-\varphi_l'(x))^2\right] < \Sigma_3' + \frac{2(s_{r+1}-s_r)}{n}(c_2\log n)^2 < \Sigma_3' + o(1)$$

$$\left(\text{since } (s_{r+1}-s_r) < c\frac{s_r}{r^{3/4}} < c\frac{s_r}{(\log n)^3}\right),$$

or for almost all x $\Sigma_3 = o(1)$. Thus we have only to prove (16).

The proof of (16) is similar to that of our lemma. Denote by $S_k'(x)$ the set in x for which $L_k(f(x)-\varphi_l(x))^2 > \varepsilon$. By (14) and the theorem of MARCINKIEWICZ—ZYGMUND[9]) the measure of $S_k'(x)$ is less than $\frac{1}{(\log k)^{10}}$. Thus from (8) if $\Sigma_3^{(s_r)} > \varepsilon s_r$, x has to lie in at least $\frac{c_5 s_r}{(\log s_r)^2}$ sets $S_k'(x)$, $1 \leqq k \leqq s_r$.

But then, as in the proof of the lemma,

$$M[x; \Sigma_3' > \varepsilon s_r] < \frac{c_6 (\log s_r)^2}{s_r} \sum_{k \leqq s_r} \frac{1}{(\log k)^{10}} < \frac{c_7}{(\log s_r)^8}$$

or

$$\sum_{r=1}^{\infty} M[x; \Sigma_3' > \varepsilon s_r] < \sum_{r=1}^{\infty} \frac{c_7}{r^2} < \infty,$$

which proves (16) and completes the proof of Theorem 2.

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